SOME OPERATIONAL CALCULUS FORMULAS FOR STEP FUNCTIONS

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One-dimensional Laplace transforms are given for some step functions.

The possibility of using operational calculus methods to obtain an exact analytical solution in a number of applied problems often depends upon finding appropriate operational formulas for step functions in handbook tables. As is well known, a function f(t), $0 \leqslant t < \infty$ is said to be a step function if the interval $(0, \infty)$ can be decomposed into a finite or denumerable number of noninteresecting intervals in each of which the function f(t) maintains a constant value [1]. Extending the existing tables of operational formulas for step functions continues to be an important task in operational calculus.

TABLE 1. Laplace Transforms of Some Step Functions

No.	$f([t])$	$F(p)$
1	$x^{[t]}$	$\frac{1-e^{-p}}{p}(1-xe^{-p})^{-1},\ \lvert xe^{-p}\rvert<1$
2	$[t]\,x^{[t]}$	$\frac{1-e^{-p}}{p}xe^{-p}(1-xe^{-p})^{-2},\ \lvert xe^{-p}\rvert<1$
3	$\frac{x^{[t]}}{([t]+a)^s}$	$\frac{1-e^{-p}}{p}\Phi(xe^{-p}, s, a),\ \lvert xe^{-p}\rvert<1,\ \operatorname{Re} a, \operatorname{Re} s>0$
4	$\frac{x^{[t]+1}}{([t]+1)^n}$	$\frac{e^p-1}{p}\operatorname{Li}_n(xe^{-p}),\ \lvert xe^{-p}\rvert<1,\ n>2$
5	$\frac{x^{[t]+1}}{[t]+1}$	$\frac{e^p-1}{p}\ln(1-xe^{-p}),\ \lvert xe^{-p}\rvert<1$
6	$\frac{x^{[t]}}{[t]!}$	$\frac{1-e^{-p}}{p}\exp(xe^{-p})$
7	$\frac{x^{2[t]}}{(2[t])!}$	$\frac{1-e^{-p}}{p}\operatorname{ch}(xe^{-p/2})$
8	$\frac{x^{2[t]+1}}{(2[t]+1)!}$	$\frac{2}{p}\operatorname{sh}\frac{p}{2}\operatorname{sh}(xe^{-p/2})$
9	$\frac{x^{2[t]+1}}{\Gamma([t]+3/2)}$	$\frac{2}{p}\operatorname{sh}\frac{p}{2}\exp(x^2e^{-p})\operatorname{erf}(xe^{-p/2})$
10	$\frac{(-1)^{[t]}}{[t]!([t]+a)}x^{[t]+a}$	$\frac{1-e^{-p}}{p}e^{ap}\gamma(a, xe^{-p})$
11	$\frac{(\pm x)^{[t]+1}}{([t]+1)!([t]+1)}$	$\frac{1-e^{p}}{p}[\mathrm{C}+\ln x-p-\operatorname{Ei}(\pm xe^{-p})]$
12	$\frac{x^{[t]+1}}{[t]!([t]+1)}$	$\frac{e^p-1}{p}[\exp(xe^{-p})-1]$
13	$\frac{x^{2[t]+1}}{[t]!(2[t]+1)}$	$\frac{\sqrt{\pi}}{p}\operatorname{sh}\frac{p}{2}\operatorname{erfi}(xe^{-p/2})$
14	$\frac{x^{2[t]+1}}{(2[t]+1)!(2[t]+1)}$	$\frac{2}{p}\operatorname{sh}\frac{p}{2}\operatorname{shi}(xe^{-p/2})$
15	$\frac{(-1)^{[t]}x^{4[t]+1}}{(2[t])!(4[t]+1)}$	$\sqrt{\frac{\pi}{2}}\frac{e^{p/4}-e^{-3p/4}}{p}C(x^2e^{-p/2})$

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TABLE 1 (Continued).

No.	$f([t])$	$F(p)$
16	$\frac{(-1)^{[t]} x^{4[t]+3}}{(2[t]+1)!\,(4[t]+3)}$	$\sqrt{\frac{\pi}{2}}\,\frac{e^{3p/4}-e^{-p/4}}{p}\,S(x^2e^{-p/2})$
17	$\frac{x^{4[t]+2}}{(2[t]+1)\,\Gamma(2[t]+3/2)}$	$\frac{\sqrt{\pi}}{p}\,\mathrm{sh}\,\frac{p}{2}\,\mathrm{erf}\,(xe^{-p/4})\,\mathrm{erfi}\,(xe^{-p/4})$
18	$\frac{x^{2[t]}}{[t]!\,\Gamma([t]+\nu+1)}$	$\frac{1-e^{-p}}{x^{\nu}p}\,e^{\nu p/2}I_{\nu}(2xe^{-p/2})$
19	$\frac{x^{[t]}}{[t]!\,\Gamma(\nu-[t]+1)}$	$\frac{1-e^{-p}}{\Gamma(\nu+1)\,p}\,(1+xe^{-p})^{\nu}$
20	$\frac{x^{2[t]}}{\Gamma([t]+3/2)\,\Gamma([t]+\nu+3/2)}$	$\frac{1-e^{-p}}{x^{\nu+1}p}\,\mathbf{L}_{\nu}(2xe^{-p/2})$
21	$\frac{\Gamma([t]/2+\nu)}{[t]!}\,x^{[t]}$	$\frac{\Gamma(2\nu)}{2^{\nu-1}}\,\frac{1-e^{-p}}{p}\,\exp\left(\frac{x^2e^{-2p}}{2}\right)D_{-2\nu}\left(-\frac{x}{\sqrt{2}}\,e^{-p}\right)$
22	$\frac{\Gamma([t]+\nu)}{[t]!}\,x^{[t]}$	$\frac{\Gamma(\nu)}{p}\,(1-e^{-p})\,(1-xe^{-p})^{-\nu},\ \lvert xe^{-p}\rvert<1$
23	$\frac{(2[t])!}{([t]!)^2}\,x^{[t]}$	$\frac{1-e^{-p}}{p}\,(1-4xe^{-p})^{-1/2},\ \lvert xe^{-p}\rvert<1/4$
24	$\frac{(2[t]+1)!}{([t]!)^2}\,x^{[t]}$	$\frac{1-e^{-p}}{p}\,(1-4xe^{-p})^{-3/2},\ \lvert xe^{-p}\rvert<1/4$
25	$\frac{(2[t])!}{([t]!)^3}\,x^{[t]}$	$\frac{1-e^{-p}}{p}\,\exp(2xe^{-p})\,I_0(2xe^{-p})$
26	$\frac{(2[t])!}{([t]!)^4}\,x^{2[t]}$	$\frac{1-e^{-p}}{p}\,I_0^2(2xe^{-p/2})$
27	$\frac{(2[t])!\,x^{2[t]}}{([t]!)^2\Gamma([t]+\nu+1)\Gamma([t]-\nu+1)}$	$\frac{1-e^{-p}}{p}\,I_{\nu}(2xe^{-p/2})\,I_{-\nu}(2xe^{-p/2})$
28	$\frac{\{(2[t])!\}^2}{([t]!)^4}\,x^{2[t]}$	$\frac{2}{\pi}\,\frac{1-e^{-p}}{p}\,\mathbf{K}(4xe^{-p/2}),\ \lvert xe^{-p/2}\rvert<1/4$
29	$\frac{\{(2[t])!\}^2}{([t]!)^4}\,\frac{x^{2[t]}}{2[t]-1}$	$\frac{2}{\pi}\,\frac{e^{-p}-1}{p}\,\mathbf{E}(4xe^{-p/2}),\ \lvert xe^{-p/2}\rvert<1/4$
30	$\left(\sum_{k=1}^{[t]+1}\frac{1}{k}\right)x^{[t]+1}$	$\frac{e^{p}-1}{p\,(xe^{-p}-1)}\,\ln(1-xe^{-p}),\ \lvert xe^{-p}\rvert<1$
31	$\left(\sum_{k=1}^{[t]+1}\frac{1}{k}\right)\frac{x^{[t]+2}}{[t]+2}$	$\frac{1-e^{-p}}{2p}\,e^{2p}\ln^2(1-xe^{-p}),\ \lvert xe^{-p}\rvert<1$
32	$\left(\sum_{k=1}^{2[t]+1}\frac{(-1)^k}{k}\right)\frac{x^{2[t]+2}}{[t]+1}$	$\frac{e^{p}-1}{p}\,\ln(1+xe^{-p/2})\,\ln(1-xe^{-p/2}),\ \lvert xe^{-p/2}\rvert<1$
33	$\left(\sum_{k=0}^{[t]}\frac{1}{2k+1}\right)\frac{x^{2[t]+2}}{[t]+1}$	$\frac{e^{p}-1}{4p}\,\ln^2\frac{1+xe^{-p/2}}{1-xe^{-p/2}},\ \lvert xe^{-p/2}\rvert<1$
34	$\left(\sum_{k=0}^{[t]}\frac{1}{2k+1}\right)\frac{(-1)^{[t]}x^{2[t]+2}}{[t]+1}$	$\frac{e^{p}-1}{p}\,\mathrm{arctg}\,(xe^{-p/2}),\ \lvert xe^{-p/2}\rvert<1$
35	$x^{[t]}\sin a[t]$	$\frac{1-e^{-p}}{p}\,\frac{xe^{-p}\sin a}{1-2xe^{-p}\cos a+x^2e^{-2p}}$
36	$x^{[t]}\cos a[t]$	$\frac{1-e^{-p}}{p}\,\frac{1-xe^{-p}\cos a}{1-2xe^{-p}\cos a+x^2e^{-2p}}$
37	$\frac{x^{[t]+1}}{[t]+1}\sin a([t]+1)$	$\frac{e^{p}-1}{p}\,\mathrm{arctg}\,\frac{xe^{-p}\sin a}{1-xe^{-p}\cos a}$
38	$\frac{x^{[t]+1}}{[t]+1}\cos a([t]+1)$	$\frac{1-e^{p}}{2p}\,\ln(1-2xe^{-p}\cos a+x^2e^{-2p})$
39	$\frac{x^{[t]}}{[t]!}\sin(a[t]+b)$	$\frac{1-e^{-p}}{p}\,\exp(xe^{-p}\cos a)\,\sin(xe^{-p}\sin a+b)$

TABLE 1 (Continued).

No.	$f([t])$	$F(p)$
40	$\frac{x^{[t]}}{[t]!}\cos(a[t]+b)$	$\frac{1-e^{-p}}{p}\exp(xe^{-p}\cos a)\cos(xe^{-p}\sin a+b)$
41	$x^{[t]},\ t<n,$ $0,\ t>n$	$\frac{1-e^{-p}}{p}\ \frac{1-x^n e^{-np}}{1-xe^{-p}}$
42	$\frac{x^{[t]}}{([t]+a)^s},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{p}[\Phi(xe^{-p},s,a)-x^n e^{-np}\Phi(xe^{-p},s,n+a)]$
43	$\frac{x^{[t]}}{[t]!},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{(n-1)!\,p}\exp(xe^{-p})\Gamma(n,xe^{-p})$
44	$\frac{x^{2[t]}}{[t]!\,(2n-2[t]-2)!},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{(2n-2)!\,p}(ix)^{2n-2}e^{-(n-1)p}H_{2n-2}\left(\frac{e^{p/2}}{2ix}\right)$
45	$\frac{x^{2[t]+1}}{[t]!\,(2n-2[t]-1)!},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{(2n-1)!\,p}(ix)^{2n-1}e^{-(n-1)p}H_{2n-1}\left(\frac{e^{p/2}}{2ix}\right)$
46	$\frac{x^{2[t]+1}}{(2[t]+1)!(n-[t]-1)!},\ t<n,$ $0,\quad t>n$	$\frac{2(-1)^{n-1}}{(2n-1)!\,ip}\operatorname{sh}\frac{p}{2}H_{2n-1}\left(\frac{ix}{2}e^{-p/2}\right)$
47	$\frac{x^{2[t]}}{(2[t])!\,(n-[t]-1)!},\ t<n,$ $0,\quad t>n$	$\frac{(-1)^{n-1}}{(2n-2)!}\ \frac{1-e^{-p}}{p}H_{2n-2}\left(\frac{ix}{2}e^{-p/2}\right)$
48	$\frac{(2n-[t]-3)!}{[t]!}x^{2[t]},\ t<n,$ $0,\quad t>n$	$\frac{2x^{2n-1}}{n-1}\ \frac{1-e^{-p}}{p}e^{-(n-1/2)p}O_{2n-2}(2xe^{-p/2})$
49	$\frac{(2n-[t]-2)!}{[t]!}x^{2[t]},\ t<n,$ $0,\quad t>n$	$\frac{4x^2}{2n-1}\ \frac{1-e^{-p}}{p}e^{-np}O_{2n-1}(2xe^{-p/2})$
50	$\frac{(n+[t]-2)!}{(2[t])!\,(n-[t]-1)!}x^{[t]},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{(n-1)\,p}T_{n-1}\left(1+\frac{x}{2}e^{-p}\right)$
51	$\frac{(n+[t]-1)!}{(2[t]+1)!\,(n-[t]-1)!}x^{[t]},\ t<n,$ $0,\quad t>n$	$\frac{1-e^{-p}}{np}U_{n-1}\left(1+\frac{x}{2}e^{-p}\right)$
52	$\binom{n}{[t]}x^{[t]},\ t<n+1,$ $0,\quad t>n+1$	$\frac{1-e^{-p}}{p}(1+xe^{-p})^n$
53	$\binom{2n-[t]}{[t]}x^{2[t]},\ t<n+1,$ $0,\quad t>n+1$	$(ix)^{2n}\frac{1-e^{-p}}{p}e^{-np}U_{2n}\left(\frac{e^{p/2}}{2ix}\right)$
54	$\binom{2n-[t]+1}{[t]}x^{2[t]},\ t<n+1,$ $0,\quad t>n+1$	$(ix)^{2n+1}\frac{1-e^{-p}}{p}e^{-(n+1/2)p}U_{2n+1}\left(\frac{e^{p/2}}{2ix}\right)$
55	$\binom{n+\nu}{n-[t]}\frac{x^{[t]}}{[t]!},\ t<n+1,$ $0,\quad t>n+1$	$\frac{1-e^{-p}}{p}L_n^{\nu}(-xe^{-p})$
56	$\binom{2n-[t]}{n}\frac{x^{[t]}}{[t]!},\ t<n+1,$ $0,\quad t>n+1$	$\frac{x^{n+1/2}}{n!\sqrt{\pi}}\ \frac{1-e^{-p}}{p}e^{-(n+1/2)p}K_{n+1/2}\left(\frac{x}{2}e^{-p}\right)$
57	$\binom{n}{[t]}^2 x^{[t]},\ t<n+1,$ $0,\quad t>n+1$	$\frac{1-e^{-p}}{p}(1-xe^{-p})^n P_n\left(\frac{e^p+x}{e^p-x}\right)$
58	$\binom{n}{[t]}\binom{[t]}{m}x^{[t]},\ t<n+1,$ $0,\quad t>n+1$	$\binom{n}{m}x^m\frac{1-e^{-p}}{p}e^{-mp}(1+xe^{-p})^{n-m}$

TABLE 1 (Continued).

No.	$f([t])$	$F(p)$
59	$\binom{n}{[t]}\binom{n+[t]}{n} x^{[t]},\ t<n+1,$ $0,\ t>n+1$	$\frac{1-e^{-p}}{p} P_n(1+2xe^{-p})$
60	$\binom{n}{[t]}^2 \binom{2n}{[t]}^{-1} x^{[t]},\ t<n+1,$ $0,\ t>n+1$	$\binom{2n}{n}^{-1} (xe^{-p})^n \frac{1-e^{-p}}{p} P_n\left(1+\frac{2}{x} e^p\right)$

Table 1 presents the relation for step functions in two columns. The left-hand column exhibits functions f([t]), where [t] is the largest integer not exceeding t; the right-hand column shows the corresponding Laplace transform F(p), where

$$F(p)=\int_0^\infty f([t])\exp(-pt)\,dt=\frac{1-e^{-p}}{p}\sum_{k=0}^{\infty} f(k)\,e^{-kp}.$$

Here $f([t]) = f(k)$ for $k \leqslant t < k+1$, $k = 0, 1, 2, \ldots$; Re $p > 0$ unless otherwise indicated. The notation employed here is that commonly appearing in the mathematical literature [3, 4].

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